An Oscillating System with Sliding Friction

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Both harmonic oscillations and friction are the types of concepts in freshman physics that are readily applicable to the "real world" and as such, most students find these ideas interesting. Damped oscillations are usually presented with resistance proportional to velocity, which has the advantage of a relatively straightforward mathematical solution. This type of resistance occurs for very slow moving bodies in fluid, although a more common resistive force in fluid is proportional to velocity squared.[1] Thus, mechanical oscillations with damping proportional to velocity may be more useful in the freshman course as an analogy for the future study of LRC circuits.[2] Whereas an oscillator with damping proportional to velocity has an exponential decay in amplitude, a system with sliding friction results in amplitude that decays in a linear manner.[3] In this paper I present a demonstration of an oscillator with sliding friction that exhibits very good agreement with a linear fall off in amplitude. The demonstration also confirms that sliding friction is proportional to the magnitude of the normal force.

Demonstration Setup

The key idea in the current setup is that only a part of the system is subject to a significant resistive force.[4] A wood friction block is placed, felt-side down, between two carts on a Dynamics Track,[5] as shown in Fig. 1. A stiff spring stretched between the carts holds the three objects together. Two additional springs attach the carts to the far sides of the track, and a motion sensor connected to the LabPro interface running the Logger Pro software[6] is placed at the end of the track to record position data. Extra weights are placed on the carts so that the total mass of the oscillating system exceeds 3 kg, which results in a longer period and makes for better data in light of a limited sampling rate of the motion sensor.

Fig. 1. Equipment setup. Note that excessive displacement in the oscillation may cause a separation between the wood friction block and the carts.

Several sets of data for the oscillations are collected. In the first trial the friction block is held slightly above the track by the two carts and the system is set in motion (Fig. 2). In the second trial, the carts are pulled apart so that the block rests on the track; then the carts are allowed to come together, making sure that only horizontal forces act on the block. The system is set in motion and data is collected (Fig. 3). In subsequent trials the process is repeated with additional small weights attached to the wood friction block (prior to the carts compressing it) to increase the normal and hence the frictional force.

Theoretical Considerations

The equation of motion for the system with sliding friction is

$$\ddot{x} = \frac{-k(x - x_0) \pm f}{m}, \qquad (1)$$

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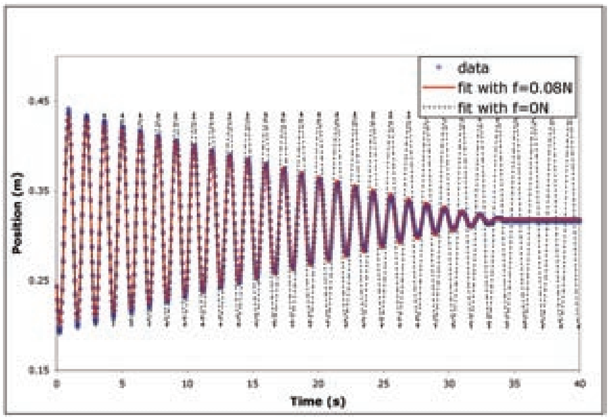

Fig. 2. Oscillations with friction block above the track.

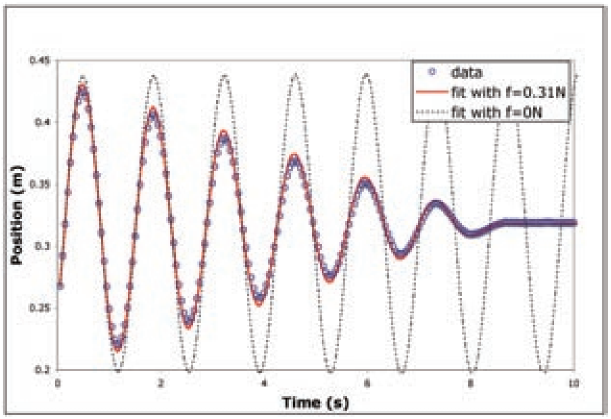

Fig. 3. Friction block in contact with the track.

where m is the mass of the system, k the effective spring constant, x_0 the equilibrium position, and $\pm f$ the constant magnitude resistive force whose sign depends on the direction of motion. Equation (1) can be solved in a straightforward albeit cumbersome manner by splitting the motion into left- and right-moving segments. Taking as the initial condition the oscillator at the right turning point with displacement A_0, we obtain[7,8]

$$x(t) = \left(A_0 - \frac{f}{k}(2n+1) \right) \cos(\omega_0 t) + (-1)^n \frac{f}{k} + x_0. \quad (2)$$

Here $\omega_0 = \sqrt{k/m}$ is the angular frequency, which does not depend on amplitude, and n counts the number of half-cycles of motion.

Linear amplitude decay can be deduced from the solution by observing 1) that the period does not change and 2) that the amplitude decreases with each half-cycle by the constant amount, $2f/k$. Note that Eq. (2) loses validity beyond the time when spring force is insufficient to overcome static friction at a turning point. Also observe that in absence of friction, i.e., in the limit of zero f, simple harmonic motion is recovered from Eq. (2), $x(t) = A_0 \cos(\omega_0 t) + x_0$.

The sliding friction force on the wood block is expected to take the form $f_b = \mu_k F_N$, where μ_k is the coefficient of kinetic friction and $F_N = m_b g$ is the magnitude of the normal force. Here m_b is the mass of the wood block plus any additional small weights attached to it.

Analysis of Data

Figure 2 contains the first trial data for the system with the friction block above the track. Students realize that the carts themselves contribute to friction since the system eventually comes to a stop, and to a good approximation, friction due to the rolling carts results in a linear fall off in oscillation amplitude. This suggests that there is a constant resistive force due to the rolling carts f_c. Hence for subsequent trials the total friction force should be parameterized as $f = f_b + f_c$, where f_b is the sliding friction force on the wood block.

Data from all trials can be fit to a linearly decaying sinusoidal function in Logger Pro. The frictional force can be extracted from the slope of oscillation amplitude decay after some analysis relating the solution in Eq. (2) to the fit function. This is a more involved process than fitting to a simple sine function in the case of no friction. So instead a procedure for solving Eq. (1) numerically is used below, since with or without friction the numerical problems are very similar and equally easily solved. The numerical solution and data can then be compared graphically to extract the frictional force directly.[9] To perform the analysis, data are exported from Logger Pro as a text file and imported into a spreadsheet like Microsoft Excel. A simple algorithm like the Euler algorithm, or a more complex one, can be used to run the simulation. The details of implementing the numerical solution are found in the appendix.

It is easiest to begin with the zero-friction numerical solution. Mass m is measured on a scale and parameters k and x_0 are adjusted to produce the correct

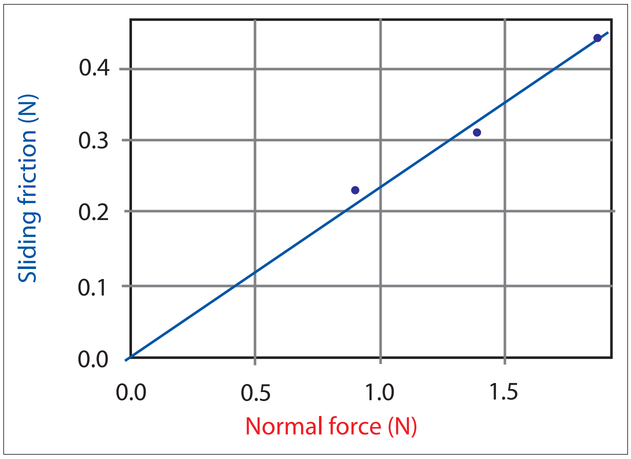

Fig. 4. Frictional force on the block.

period and offset when compared to data graphically. Next the effect of friction is included to produce a good fit to the decaying amplitude. To implement the sign in the frictional force in Eq. (1), it is useful to parameterize it by $\mp f = -v/|v|f$. Note also that the values of k and x_0 from the first trial can be used in the subsequent trials but that the mass of the oscillating system changes in later trials due to additional small weights on the friction block.

Figure 2 contains position data for the first trial, a fit to data with $f = 0$, and a good fit to data with $f = 0.08$ N. This confirms that friction of rolling carts can be parameterized by a constant force, $f_c = 0.08$ N. Note also, that the solution with and without friction results in identical periods as anticipated.

Figure 3 contains position data for the second trial, a fit to data with $f = 0$, and a good fit to data with $f = 0.31$ N. Sliding frictional force on the wood block is isolated by the subtraction $f_b = f - f_c$.

A nice way to present the results of this experiment is to graph the sliding frictional force on the block versus the normal force on the block, as shown in Fig. 4. The graph by its linear nature confirms the relation $f_b = \mu_k F_N$ for the given range of normal force. The averaged coefficient of sliding friction is extracted from the slope of Fig. 4, $\mu_k = 0.23$.

Conclusion

The demonstration presented here shows 1) that frictional force on rolling carts is approximately constant, 2) that sliding friction in oscillations results in a linear amplitude fall off, 3) that sliding frictional force is constant, and 4) that over the range of normal force tested, the sliding frictional force is proportional to the normal force on the block. The demonstration can be adapted as a laboratory exercise with some modifications, depending on the length of the laboratory period and the experience of students in performing similar analyses. Additionally, one can imagine using the procedure to study sliding friction for various surfaces by covering the track between the carts and the wood block with different materials.

Appendix

The numerical solution to Eq. (1) is presented below. To solve the second-order Eq. (1), we begin by rewriting it as a system of first-order equations for position and velocity:

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= \frac{-k(x-x_0) \mp f}{m}. \end{aligned} \tag{3}$$

Next the system in Eq. (3) is solved numerically in a spreadsheet with the following step-by-step procedure:

$$\begin{aligned} x(t+dt) &\approx x\big|_t + v\big|_t\, dt + \frac{1}{2}\frac{dv}{dt}\bigg|_t dt^2 \\ v(t+dt) &\approx v\big|_t + \frac{dv}{dt}\bigg|_t dt + \frac{1}{2}\frac{d^2v}{dt^2}\bigg|_t dt^2. \end{aligned} \tag{4}$$

Equation (4) is the second-order Taylor series method for numerically solving differential equations.[10] Without the dt^2 terms, this would be the so-called Euler method, which can be used in our oscillatory problem provided the time step is taken sufficiently short (at least 100 times shorter than the time step from our data). This suggests that it may be simpler to increase the accuracy of the solution method by taking more terms in the Taylor series, as in Eq. (4), so that the numerical solution and data share the same time step dt.

In Eq. (4) the derivatives are evaluated numerically using the expressions in Eq. (3). The formula for the last derivative is obtained by differentiating the second expression in Eq. (3):

$$\frac{d^2v}{dt^2} = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d}{dt}\left(\frac{-k(x-x_0) \mp f}{m}\right) = \frac{-kv}{m}. \tag{5}$$

If more terms are desired in Eq. (4), formulas for higher derivatives are obtained in the same way as in Eq. (5).

References

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10. See for instance Section 8.8 of George Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic Press, 1985); or an introductory text on ordinary differential equations.

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