Coupled oscillations in suspended magnets

Michael J. Moloney[a)]

Department of Physics and Optical Engineering, Rose-Hulman Institute of Technology, Terre Haute, Indiana 47803

A pair of strong magnets suspended by a single thread between them can act as a pendulum and also exhibit torsion as the magnets align with the Earth's magnetic field. This arrangement demonstrates visible coupled oscillations, alternating between pendulum motion and twisting about the line of the thread, often at large angles. The vertical component of the Earth's magnetic field is the coupling agent between the pendulum and twisting motions. The pendulum and torsion motions become uncoupled if an external coil is used to cancel this field component. An analysis of this motion is in agreement with experimental observations. © *2008 American Association of Physics Teachers.*
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I. INTRODUCTION

A pair of magnets hanging by a single thread (see Fig. 1) can oscillate like a pendulum, but may also oscillate in a twisting mode, where the magnetic moment attempts to point toward the Earth's north pole (North). The potential energy of this system includes a dependence on the angle between the magnets' magnetic moment and the Earth's magnetic field. This dependence leads to a coupling between the torsional and pendulum motions through B_v, the Earth's vertical component of magnetic field. For a pair of small, strong magnets, the coupled motion is easily observed and alternates between pendulum and torsional motion (see Fig. 2).

The parameters of this motion may be changed by varying the pendulum length and the vertical or horizontal component of the magnetic field using an external coil oriented in either the horizontal or vertical plane. When the torsional angles are kept small, one can obtain analytical expressions for the normal modes. For large torsional angles, modeling may be done by numerical integration of the equations of motion.

II. EXPERIMENTS

A pair of disk-shaped magnets[1] was used, each 3.0 mm thick (see Fig. 1). Each magnet had an 18 mm diameter and a mass of 11.55 g. The magnetic moment μ of the two magnets was found to be $\mu=1.44$ A m^2 by the method of Ref. 2. This method involves hanging the magnets at the center of a coil whose plane coincides with the plane of the magnets when they point North. The torsional period was measured for several currents carried by the coil, and a plot was made of the torsional frequency squared versus the coil current. From this plot, both μ and the Earth's horizontal magnetic field component B_h were determined. We take L to be the distance from the support to the magnets' center of mass, $D=6.0$ mm to be the total thickness of the magnet pair, and $R=9$ mm to be the radius of either magnet.

For L around 0.2 m one could let the thread hang vertically and observe almost pure torsional oscillations with no observable coupled motion. From the small angle motion of the magnet pair, a torsional frequency of $f_\tau=1.61$ Hz was observed. The horizontal component of the Earth's magnetic field B_h was determined from $f_\tau=1/(2\pi)\sqrt{(\mu B_h/I_\tau)}$, where I_τ is the moment of inertia of the magnets for torsional motion. The result was $B_h=1.90\times10^{-5}$ T at the location where all

the magnet tests were done. (If the value of μ were not known, we would determine the quantities μB_h and μB_v, without finding B_h and B_v.)

For coupled motion it was convenient to measure T_{p-p}, the interval between successive times when the magnets are most nearly acting like a pendulum, with little or no visible torsion. This time was measured for a series of L values as shown in Table I. Each observation involved ten periods and was done manually with a stopwatch. Video capture and magnetic field detection are both feasible alternatives for measuring T_{p-p}.

Four methods were employed to estimate B_v. A "Magnaprobe"[3] with two gimballed axes showed the Earth's field to be approximately 45° below the horizontal at the magnets' location. This orientation implied that $B_v \cong B_h \approx 1.9\times10^{-5}$ T. Next, a coil from a commercial e/m apparatus (320 turns, average diameter 136 mm, thickness 17.5 mm) was oriented with its plane horizontal. For a suitable current it could be made to cancel B_v and uncouple the magnet oscillations. This arrangement worked as intended, but the exact coil current at which the oscillations were uncoupled was difficult to establish. Run-to-run variations in the current were about 10–15%.

The time T_{p-p} was calculated from theory [see Eq. (10)] and compared to the measured values in Table I. B_h was considered to have a known value of 1.90×10^{-5} T. Then B_v was adjusted to seek the best agreement with the T_{p-p} values in Table I. This procedure gave a B_v value of 2.22×10^{-5} T, which is taken as the most precise value of B_v for the location where the magnets were suspended.

The fourth method for estimating B_v involved measuring the angle of the plane of the magnets from the vertical, using a horizontal laser beam. The magnetic moment μ of the suspended magnets pointed at a slight angle α below the horizontal due to the equality of the torque from B_v and the gravitational torque:

$$mg\,R\sin\alpha = \mu B_v \sin(\pi/2-\alpha). \tag{1}$$

In Eq. (1) α is small, $\sin(\pi/2-\alpha)$ is nearly 1, and $\alpha \cong \mu B_v/(mg\,R)$. With $B_v=2.22\times10^{-5}$ T, and $\mu=1.44$ A m^2, we find α to be about 1.9°. The angle α was roughly measured using a level laser beam reflected from the magnets. Because the reflected beam was too diffuse, small rectangular glass mirrors (each 0.8 g, 11.6 mm on a side, 2 mm thick, of the type sold for hobby applications) were attached to each side of the magnet pair. Adding the little mirrors gave a

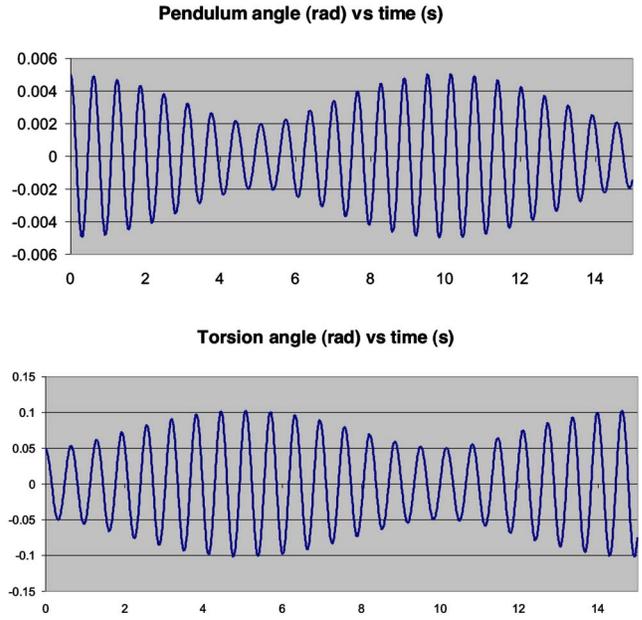Fig. 1. Two small, strong magnets suspended by a thread between them.

Fig. 2. Numerical integration of Eqs. (6) and (7) for $L = 0.10$ m. (The pendulum angles are small but torsional angles are not.)

reflected beam precise enough to determine that α was about $1.7°$. Adding the mirrors' masses changed the mass m in Eq. (1) and reduced the calculated α value to about $1.6°$, in good agreement with its measured value.

The angle $\gamma = \tan^{-1}(B_v/B_h) = \tan^{-1}(2.22 \times 10^{-5}/1.90 \times 10^{-5}) = 49.5°$ agrees roughly with the observed Magnaprobe dip angle in the room where the experiment was done, but disagrees with National Oceanic and Atmospheric Administration magnetic field information[6] for Terre Haute, where $\gamma \cong 70°$. A survey of Magnaprobe dip angles was done in several open areas on the Rose-Hulman campus, and values of $70°$ and even $80°$ were routinely observed. It is likely that the values of B_v and B_h reported here are local values, and would be different if the experiment were repeated in another location in the same building. Because indoor values of B_h and B_v are likely to vary significantly with position, it is advisable to carry out all measurements with the magnets in exactly the same spot.

III. ANALYSIS

The x–y plane was selected as the plane of pendulum oscillations. The z axis was chosen to point in the direction (North) of the magnet's dipole moment $\boldsymbol{\mu}$ when it is at rest. The moment μ dipped about $\alpha = 2°$ below the horizontal, but in the following analysis, the slight dip angle is ignored and the motion of the magnet is treated as if $\alpha = 0$. The pendulum makes an angle θ to the vertical (y axis pointing downward), and the magnets' torsion angle is ψ about the line from the center of mass to the pendulum support.

Table I. Comparison of theoretical and experimental small angle T_{p-p} for various pendulum lengths L, with $B_v = 2.22 \times 10^{-5}$ T. The theory [Eq. (10)] and numerical integration of Eqs. (8) and (9) give identical answers.

Length L (mm)	T_{p-p} (exp) (s)	T_{p-p} [Eq. (12)] (s)
70	3.4	3.5
85	6.5	7.1
92	9.2	9.8
108	7.6	7.8
128	4.2	4.4
154	2.9	2.9
176	2.4	2.4

The magnets' potential energy is $U_m = -\boldsymbol{\mu} \cdot \mathbf{B}$. To find U_m we express the Earth's field as $\mathbf{B} = (0, B_v, B_h)$, and the magnetic moment as $\boldsymbol{\mu} = \mu (\sin \psi \cos \theta, \sin \psi \sin \theta, \cos \psi)$. Then

$$U_m = -\mu B_v \sin \psi \sin \theta - \mu B_h \cos \psi. \qquad (2)$$

The pendulum's potential energy for a mass m and length L is $U_p = mg L (1 - \cos \theta)$. The Lagrangian is

$$L = 1/2 I_p \dot{\theta}^2 + 1/2 I_\tau \dot{\psi}^2 - mg L(1 - \cos \theta)$$
$$+ \mu B_v \sin \psi \sin \theta + \mu B_h \cos \psi \qquad (3)$$

with $I_p = mL^2 + 1/2mR^2$, and $I_\tau = 1/4mR^2 + 1/12mD^2$.

Lagrange's equations of motion are

$$I_\tau \ddot{\psi} = -\mu B_h \sin \psi + \mu B_v \cos \psi \sin \theta \qquad (4)$$

and

$$I_p \ddot{\theta} = -mg L \sin \theta + \mu B_v \sin \psi \cos \theta. \qquad (5)$$

In the coupled oscillations of the magnet pair, torsional angles exceeding $90°$ are fairly common, and easy to understand in terms of energy. A magnet pair with $L = 0.10$ m at an angle of $5°$ to the vertical has a gravitational potential energy $U_g = 43 \times 10^{-6}$ J more than when the pair is hanging vertically. A magnet pair oriented $90°$ to North has a magnetic potential energy $U_m = 28 \times 10^{-6}$ J greater than when it is aligned with the Earth's magnetic field. A magnet pair launched pendulum style at a modest $5°$ angle to the vertical will later be hanging nearly vertically with almost all its energy in torsional motion. Most of the initial U_g on launch will be present in the torsional motion, generating torsional angles of $90°$ or more.

Equations (4) and (5) may be written as

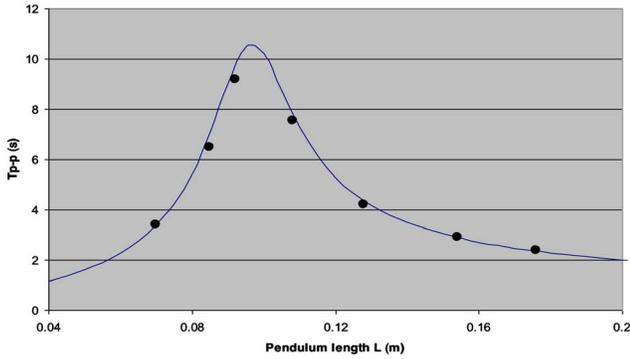$$\ddot{\psi} = -a \sin \psi + \alpha b \cos \psi \sin \theta \qquad (6)$$

and

Fig. 3. Small angle T_{p-p} versus pendulum length. The dots are experimental values, and the curve is calculated from Eq. (12) using $B_v = 2.22 \times 10^{-5}$ T.

$$\ddot{\theta} = -c \sin\theta + b \sin\psi \cos\theta, \tag{7}$$

where $a = \omega_\tau^2 = \mu B_h / I_\tau$, $b = \mu B_v / I_p$, $c = \omega_p^2 = mg L / I_p$, and $\alpha = I_p / I_\tau$; ω_τ is the torsion frequency in the absence of coupling, and ω_p is the corresponding pendulum frequency. Equations (6) and (7) were integrated numerically (using second-order Runge-Kutta[4] for 4000 time steps of 4 ms or less) in a spreadsheet. Typical results for ψ and θ as a function of time are shown in Fig. 2. The numerical integration handled both large and small torsional angles smoothly.

For small angles in Eqs. (6) and (7) we have

$$\ddot{\psi} = -a\psi + \alpha b\theta \tag{8}$$

and

$$\ddot{\theta} = -c\theta + b\psi. \tag{9}$$

The coupling term in Eq. (3) for small angles is $\mu B_v \psi \theta$, a form referred to in Ref. 5 as "linear coupling." Equations (8) and (9) are the analogs of Eqs. (2) and (3) in Ref. 5. We can solve for the normal-mode oscillation frequencies from Eqs. (8) and (9) as was done in Ref. 5 by eliminating ψ between them (θ could just as well have been eliminated). The resulting normal-mode frequencies are

$$\omega_\pm^2 = \left[\frac{a+c}{2} \right] (1 \pm \sqrt{(1+e)}), \tag{10}$$

where

$$e = \frac{4(\alpha b^2 - ac)}{(a+c)^2}. \tag{11}$$

The time interval T_{p-p} may be expressed for small-angle motion as

$$T_{p-p} = \frac{2\pi}{\omega_+ - \omega_-} \quad (\text{small angles } \theta \text{ and } \psi). \tag{12}$$

As noted, it is difficult to launch the magnets without producing large torsional angles ψ. A flat sheet of aluminum near the magnets will reduce their oscillations through eddy current damping, so a sheet may be carefully brought near the magnets to reduce the torsional angles to a desirable level, then gently removed.

A series of small-angle times was measured for different values of L, and the results are given in Table I. Figure 3 plots the experimental and calculated T_{p-p} values versus the pendulum length L, using $B_v = 2.22 \times 10^{-5}$ T. A sharp maximum occurs very close to where the torsional and pendulum frequencies are equal ($a=c$). If $a=c$, e becomes

$$e = \alpha(b/c)^2 - 1. \tag{13}$$

For our experiments, $\alpha(b/c)^2 \ll 1$. With this condition, and $L^2 \gg R^2$ and some algebra we find

$$T_{p-p,\max} \cong \frac{B_h}{B_v} \frac{1}{f_\tau} \frac{2L}{\sqrt{R^2 + D^2/3}}. \tag{14}$$

To match the pendulum and torsional frequencies ($a=c$), we must have $L=96$ mm. This L value was used to test the approximation in Eq. (14). It gave $T_{p-p,\max} = 10.6$ s, in good agreement with Fig. 3.

Equation (12) was compared to the numerical integration for small angles θ and ψ and found to agree perfectly. A fast Fourier transform (FFT) using Logger Pro[7] was done on the small-angle simulation results. Within the somewhat low resolution available from this data set, Eq. (10) was verified and showed the two normal modes to be approximately 0.95 and 1.64 Hz.

For a pendulum length of 98 mm, close to where a maximum is expected for T_{p-p}, magnetic field data were taken with a magnetic field probe in Logger Pro[7] for small-angle motions, with the detector on the axis of the stationary magnets 105 mm away, and oriented to observe the magnets' tangential component of B. Data were taken for 120 s at a sampling rate of 50 Hz. The FFT had a resolution of 0.013 Hz, and gave peaks at about 1.55 and 1.65 Hz, in comparison to calculated values from Eq. (10) of 1.550 and 1.644 Hz, respectively.

For larger launch angles θ and torsional angles ψ of 60° or more, T_{p-p} was strongly dependent on ψ. It was difficult to determine ψ precisely, but the numerical integration depended on ψ in approximately the same way as the observed T_{p-p}.

For $\theta=0$, Eq. (5) still indicates coupling between θ and ψ. This coupling implies that magnets hanging perfectly still and released from an initial deflection angle ψ_o will develop oscillations in θ. Integration of Eqs. (4) and (5) showed that for $\psi_o = 90°$ and the magnets initially still, small pendulum oscillations of amplitude 1–2° in θ would occur for $L = 85$ mm. This predicted behavior was checked by hanging the magnets in the center of a pair of energized Helmholtz coils, such that the magnets had an initial deflection angle ψ_o around 100° from North. When the Helmholtz current was switched off with the magnets perfectly still, the magnets were seen to deflect some 2–2.5 mm to either side in the subsequent coupled motions. Because 2/85 radians is about 1.5°, this result is in semiquantitative agreement with the numerical integration.

IV. SUMMARY

The coupled oscillations of suspended strong magnets are well described by a nonlinear coupling. For large torsional angles ψ, the transfer time between modes depends on ψ. For small torsional angles, the exact normal mode frequencies are calculated and agree with experiment.

The coupling agent is B_v, the Earth's vertical component of magnetic field. The B_v value of 2.22×10^{-5} T is a "local" value determined from the best fit of the measured T_{p-p} values. It agrees with the rough value found from uncoupling

the magnet motion using a horizontal coil, and it implies that the Earth's field is some 50° below the horizontal, close to the Magnaprobe estimate. This B_v value is also consistent with the observed angle α of the magnets' magnetic moment μ below the horizontal.

Students might begin with a pair of magnets and use the method of Ref. 2 to find μ and B_h. An alternative would be to first use a coil in the vertical plane to cancel B_h, as judged by the suspended magnets wandering around instead of oscillating. The pure torsional frequency can be measured for a long pendulum length L, so that μ could be calculated from it knowing B_h.

B_v can then be obtained in one or more of the following ways. The angle α below the horizontal can be measured with small mirrors attached to the magnets, using a level laser beam (keeping in mind the beam deflection is 2α) and $B_v = mg\,R\alpha/\mu$. With this same setup and a coil mounted in the horizontal plane, we can find the current needed to cancel B_v and reduce α to zero. With this same coil orientation we can determine the current needed to cancel B_v, as judged by the uncoupling of the torsional and pendulum oscillations of the suspended magnets.

With the values of B_h, B_v, and μ established, students can measure the small angle T_{p-p} for different values of L, and compare their results to Eqs. (10)–(12). The maximum value of T_{p-p} can be checked, and its dependence on B_v in Eq. (14) tested. The length L for $T_{p-p,\text{max}}$ would stay the same, and a coil would strengthen or weaken B_v. Another set of T_{p-p} predictions can be made with altered rotational properties (for example, by adding nonmagnetic masses like small glass mirrors to the magnets' faces), and checked by measurements.

Magnets hanging perfectly still can be positioned at various angles ψ_o to North using energized Helmholtz coils and then released when the current is turned off. With careful video analysis, the small angle pendulum motions in θ can be compared to the solutions of Eqs. (6) and (7). It might also be possible to use magnetic field detection to infer the θ amplitude, based on the changing distance to the detector from the magnets (treated as a point magnetic dipole).

Students planning to undertake magnetic field measurements of T_{p-p} should be encouraged to understand which component of B from the magnets (radial or tangential) will be most sensitive to changes in θ or the torsional angle ψ at a particular detector location.

[a] Electronic mail: moloney@rose-hulman.edu

[1] Neodymium Super Magnets, The Magnet Source, ⟨www.magnetsource.com⟩. The author purchased his magnets at a local hardware store.

[2] Michael J. Moloney, "Small strong magnets," Phys. Teach. **45**(9), 352–355 (2007).

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[4] C. W. Misner and P. Cooney, *Spreadsheet Physics* (Addison-Wesley, New York, 1991), Sec. 6.4.

[5] Richard E. Berg and Todd S. Marshall, "Wilberforce pendulum oscillations and normal modes," Am. J. Phys. **59**, 32–38 (1991).

[6] Magnetic field values at different locations and times may be found at ⟨ngdc.noaa.gov/seg/geomag/jsp/IGRFWMM.jsp⟩.

[7] Items LABPRO and MG-BTA are available from Vernier Software, ⟨www.vernier.com⟩.

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